The Circle - The Conics

- $S \equiv x^2 + y^2 - r^2$ (circle)
  $S \equiv y^2 - 4ax$ (parabola)
  $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, $a > b$ (ellipse)
  $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ (hyperbola)
- $S_1 = 0$ for circle: $xx_1 + yy_1 - r^2 = 0$
  For parabola: $yy_1 - 2ax - 2ax_1 = 0$
  For ellipse: $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$
  For hyperbola: $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$
  If $y = mx + c$ is a tangent, then
  For circle: $c = \pm r\sqrt{1 + m^2}$
  For parabola: $c = \frac{a}{m}$
  For ellipse: $c = \pm\sqrt{a^2m^2 + b^2}$
  For hyperbola: $c = \pm\sqrt{a^2m^2 - b^2}$
- **Tangent in slope form**
  For circle: $y = mx \pm r\sqrt{1 + m^2}$
  For parabola: $y = mx + \frac{a}{m}$
  For ellipse: $y = mx \pm \sqrt{a^2m^2 + b^2}$
  For hyperbola: $y = mx \pm \sqrt{a^2m^2 - b^2}$
- Quadratic in 'm' at (x_1, y_1):
  For circle:
  $m^2(x_1^2 - r^2) - 2x_1y_1m + (y_1^2 - r^2) = 0$
  For parabola: $m^2x_1 - my_1 + a = 0$
  For ellipse:
  $m^2(x_1^2 - a^2) - 2x_1y_1m + (y_1^2 - b^2) = 0$
  For hyperbola:
  $m^2(x_1^2 - a^2) - 2x_1y_1m + (y_1^2 + b^2) = 0$
- For circle: $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - r^2}$;
  $m_1m_2 = \frac{y_1^2 - r^2}{x_1^2 - r^2}$
  For parabola: $m_1 + m_2 = \frac{y_1}{x_1}$; $m_1m_2 = \frac{a}{x_1}$
  For ellipse: $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$;
  $m_1m_2 = \frac{y_1^2 - b^2}{x_1^2 - a^2}$
  For hyperbola: $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$;
  $m_1m_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2}$
- For $\perp$r tangents ($m_1m_2 = -1$), locus equation:
  For circle: $x^2 + y^2 = 2r^2$
  For parabola: $x + a = 0$
  For ellipse: $x^2 + y^2 = a^2 + b^2$
  For hyperbola: $x^2 + y^2 = a^2 - b^2$
- Point of contact of $y = mx + c$ with
  Circle: $\left(\frac{-mr^2}{c}, \frac{r^2}{c}\right)$
  Parabola: $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
  Ellipse: $\left(\frac{-ma^2}{c}, \frac{b^2}{c}\right)$
  Hyperbola: $\left(\frac{-ma^2}{c}, \frac{-b^2}{c}\right)$

- **$S_1 = S_{11}$ for**
  Circle: $xx_1 + yy_1 = x_1^2 + y_1^2$
  Parabola: $yy_1 - 2ax = y_1^2 - 2ax_1$
  Ellipse: $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$
  Hyperbola: $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$
- **Parameter point:**
  Circle: $(r\cos\theta, r\sin\theta)$
  Parabola: $(at^2, 2at)$
  Ellipse: $(a\cos\theta, b\sin\theta)$
  Hyperbola: $(a\sec\theta, b\tan\theta)$
- **Tangent at parameter:**
  Circle: $x\cos\theta + y\sin\theta = r$
  Parabola: $yt = x + at^2$
  Ellipse: $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$
  Hyperbola: $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$
- **Normal at parameter:**
  Circle: $x\sin\theta - y\cos\theta = 0$
  Parabola: $y + xt = 2at + at^3$
  Ellipse: $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$
  Hyperbola: $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$
- If the circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g_1x + 2f_1y = 0$ touch each other, then show that $gf_1 = g_1f$.

Ans: $C_1 = (-g, -f)$; $C_2 = (-g_1, -f_1)$

$r_1 = \sqrt{g^2 + f^2}$; $r_2 = \sqrt{g_1^2 + f_1^2}$

When the circles touch each other, we have

$C_1C_2 = |r_1 \pm r_2|$

$\Rightarrow C_1C_2^2 = (r_1 \pm r_2)^2$

$\Rightarrow (g - g_1)^2 + (f - f_1)^2$

$= \left(\sqrt{g^2 + f^2} \pm \sqrt{g_1^2 + f_1^2}\right)^2$

$\Rightarrow g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1$

$= g^2 + f^2 + g_1^2 + f_1^2 \pm 2\sqrt{(g^2 + f^2)(g_1^2 + f_1^2)}$

$= -2(gg_1 + ff_1)$

$= \pm 2\sqrt{g^2g_1^2 + f^2f_1^2 + g^2f_1^2 + g_1^2f^2}$

$\Rightarrow (gg_1 + ff_1)^2 = g^2g_1^2 + f^2f_1^2 + g^2f_1^2 + g_1^2f^2$

$\Rightarrow 2gg_1ff_1 = g^2f_1^2 + g_1^2f^2$

$\Rightarrow (gf_1 - fg_1)^2 = 0$

$\Rightarrow gf_1 = g_1f$

$\therefore$ Proved the result.

- Prove that the common tangents drawn to the parabola $y^2 = 16x$ and the circle $x^2 + y^2 = 8$ are given by the equations $y = \pm(x + 4)$.

Ans: $y^2 = 16x \Rightarrow a = 4$; $x^2 + y^2 = 8 \Rightarrow r = 2\sqrt{2}$

Any tangent to the parabola takes the form

$y = mx + \frac{4}{m}$ (or) $m^2x - my + 4 = 0$

This is a tangent to the given circle implies

$2\sqrt{2} = \left|\frac{0 - 0 + 4}{\sqrt{m^4 + m^2}}\right|$

$\Rightarrow 1 = \left|\frac{\sqrt{2}}{\sqrt{m^4 + m^2}}\right|$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow m^2 - 1 = 0$ $[\because m^2 + 2 \neq 0]$

$\Rightarrow m = \pm 1$

$\therefore$ The tangents are $y = \pm x \pm 4$

But only $y = x + 4$ and $y = -x - 4$ are satisfied by the condition $c = \frac{a}{m}$

$\therefore$ The common tangents are given as

$y = \pm(x + 4)$

- Find the equation to the locus of all middle points of the chords of the parabola $y^2 = 4ax$ and subtending a right angle at its vertex.

Ans: Let M (x_1, y_1) be any variable mid point of the chord of the parabola $y^2 = 4ax$, then we have

$S_1 = S_{11}$

$\Rightarrow yy_1 - 2ax = y_1^2 - 2ax_1$

$\Rightarrow \frac{yy_1 - 2ax}{y_1^2 - 2ax_1} = 1$

These chords subtend a right angle at the vertex. This implies that

$y^2 - 4ax(1) = 0$

$\Rightarrow y^2 - 4ax\left(\frac{yy_1 - 2ax}{y_1^2 - 2ax_1}\right) = 0$

$\Rightarrow y^2(y_1^2 - 2ax_1) - 4axyy_1 + 8a^2x^2 = 0$

For a right angle, x^2 Coef. + y^2 Coef. = 0

$\Rightarrow y_1^2 - 2ax_1 + 8a^2 = 0$

$\Rightarrow y^2 - 2ax + 8a^2 = 0$ is the required equation of the locus.

- Find the equation to the chord joining the points P (α) and Q (β) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for $a > b$.

Ans: For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, we have

P (α) = P $(a\cos\alpha, b\sin\alpha)$ and

Q (β) = Q $(a\cos\beta, b\sin\beta)$

Now, the equation of the chord PQ is:

$\frac{b\sin\beta - b\sin\alpha}{a\cos\beta - a\cos\alpha} = \frac{y - b\sin\alpha}{x - a\cos\alpha}$

$\frac{b\left(2\cos\frac{\beta + \alpha}{2}\sin\frac{\beta - \alpha}{2}\right)}{a\left(-2\sin\frac{\beta + \alpha}{2}\sin\frac{\beta - \alpha}{2}\right)} = \frac{y - b\sin\alpha}{x - a\cos\alpha}$

$\Rightarrow b\left(\cos\frac{\beta + \alpha}{2}\right)(x - a\cos\alpha)$

$= -a\left(\sin\frac{\beta + \alpha}{2}\right)(y - b\sin\alpha)$

$\Rightarrow bx\cos\frac{\beta + \alpha}{2} + ay\sin\frac{\beta + \alpha}{2}$

$= ab\left[\sin\alpha\sin\frac{\beta + \alpha}{2} + \cos\alpha\cos\frac{\beta + \alpha}{2}\right]$

$\Rightarrow bx\cos\frac{\alpha + \beta}{2} + ay\sin\frac{\alpha + \beta}{2}$

$= ab.\cos\frac{\alpha - \beta}{2}$

$\Rightarrow \frac{x}{a}\cos\frac{\alpha + \beta}{2} + \frac{y}{b}\sin\frac{\alpha + \beta}{2}$

$= \cos\frac{\alpha - \beta}{2}$ is the required equation.

- If the normal at one end of the latus rectum of an ellipse passes through the other end of the minor axis, then prove that $e^4 + e^2 = 1$.

Ans: Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$

Let P $(ae, b^2/a)$ be one end of the latus rectum and let Q $(0, -b)$ be the other end of the minor axis.

Normal at P is given by

$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2$

This passes thro' Q $(0, -b)$ implies

$0 - a(-b) = a^2 - b^2 = a^2e^2$

$\left[\because e^2 = \frac{a^2 - b^2}{a^2}\right]$

$\Rightarrow ab = a^2e^2$

$\Rightarrow a^2b^2 = a^4e^4$

$\Rightarrow b^2 = a^2e^4$

$\Rightarrow a^2(1 - e^2) = a^2e^4$

$\Rightarrow 1 - e^2 = e^4$

(or) $e^4 + e^2 = 1$

$\therefore$ The result proved.